Introduction to Phase Utilization with Coherent Pulsed Doppler Radars

The *Samraksh* Company
December 25, 2008

Typical non-coherent Pulsed Doppler Radars (PDRs) report only the in-phase component of the Doppler frequency, in contrast coherent PDRs report the full complex Doppler frequency. In high signal-to-noise environments it is possible to use either type of PDR to measure target motion on a scale of fractions of a wavelength. However, the method for doing this is fairly different for coherent the two types of radars, and the methods for coherent PDRs tend to be far more robust with respect to noise.

Specifically, for coherent PDRs when the return is dominated by one target it is advantageous to use changes in the phase of the Doppler frequency as a direct measure of target displacement. The Doppler frequency (which corresponds to target velocity) is equivalent to the rate of change of the Doppler phase, which implies that phase corresponds to target displacement from an arbitrary reference. Thus tracking phase changes is equivalent to tracking target displacements. Because the phase can be measured to fractions of a wavelength, this is a very fine resolution method.

Because tracking phase changes can be made relatively robust with respect to noise, it is often considered practical to use a coherent PDR to extract fine-scale motion information and impractical to use a non-coherent PDR for the same types measurements. However, theoretically similar information can be obtained from a non-coherent PDR.

This document is a tutorial on the methods of using phase information, naturally produced by a coherent PDR to make fine-scale motion measurements.

The Functional Operation of PDRs

PDRs generate pulses similar to the one shown in Figure 1. This pulse travels to the target, is reflected back, and is then compared to a delayed copy of the pulse. This is illustrated in Figure 2.

In a high power, high cost PDRs it is conceptually possible to compare the return signal to the reference signal using a wide range of numerical methods, such as non-linear impulse response inversion; however, for low power, low cost PDRs the comparison method is typically some variant of the correlation between the returned signal and the reference signal. When the correlation is high the match is good. The autocorrelation of the signal shown in Figure 1 is shown in Figure 3.

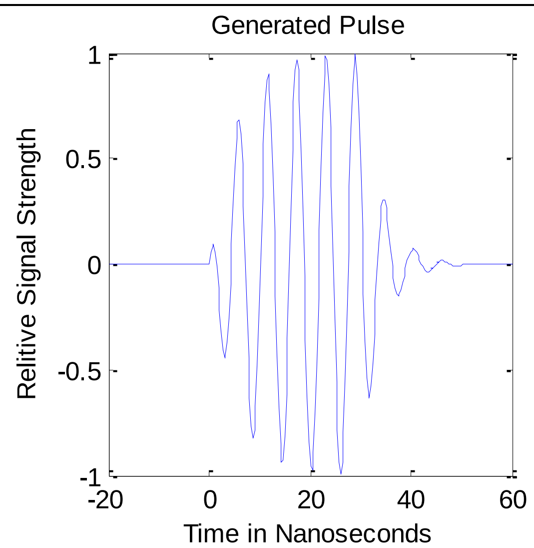

Figure 1. A stylized pulse of the type that are used by PDRs.

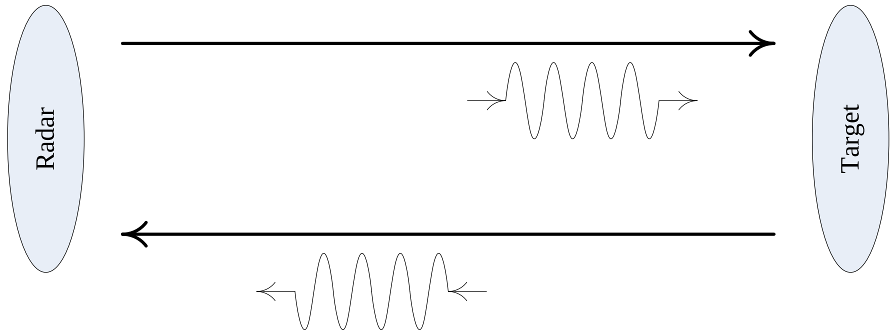

Figure 2. Stylized pulse flight.

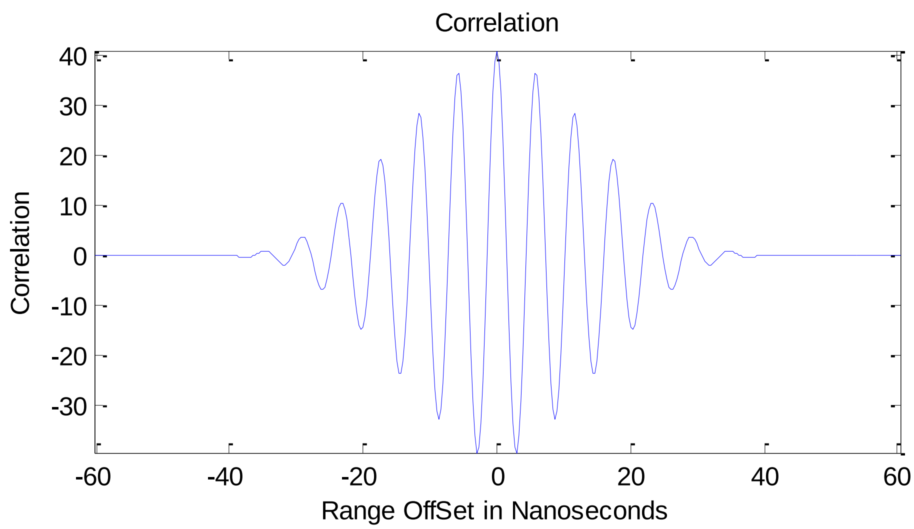

Figure 3. The autocorrelation of the signal shown in Figure 1.

For our purposes the most significant feature in Figure 3 is that the response as a function of flight path length is approximately periodic in the wavelength. For the correlation function the result is less selective in range than the original pulse and the amplitude various somewhat significantly over a reasonably chosen window of detectability. For these reasons the *BubbleBee* doesn't use the simple correlation, instead it uses a comparison method that is only slightly more complex, but produces a response that is: 1) periodic, with period of the wavelength, over a window of detectability, 2) has a nearly constant response envelope as a function of range over the window of detectability, and 3) has a window of detectability that is nearly the same width as the pulse.

Let r denote the difference between actual flight path length and the delay between the transmitted pulse and the reference pulse, let w denote the width of the pulse, and let O denote the output of the PDR. Then a very useful model of the BumbleBee is that,

$$O(r) = \begin{cases} A\cos\left(\dfrac{r - w/2}{\lambda}\right) & 0 \le r \le w \\ 0 & Otherwise \end{cases},$$

where A depends only on the target's Radar Cross Section (RCS).

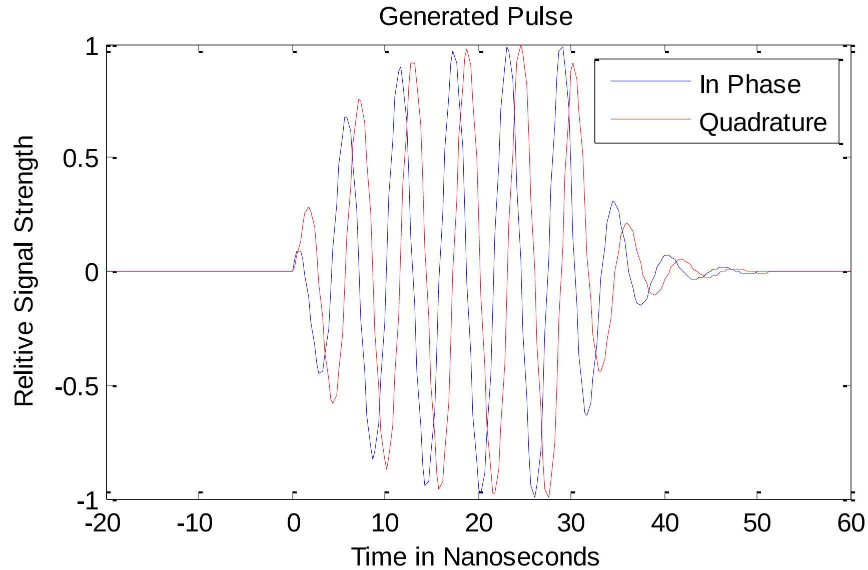

Figure 4. One of the pairs of reference waveforms that might correspond to the waveform in Figure 1.

The Operation of a Coherent PDR

A coherent radar will compare the return to two different reference signals that are 90 degrees out of phase. Example reference signals, generated from the reference signal in Figure 1, are shown in Figure 4. The correlation between the transmitted waveform and the two reference waveforms is shown in Figure 5.

If we imagine that the correlation responses are more or less steady state constant frequency oscillations, then it is natural to adopt the same phasor notation that is used for steady state electrical circuit analysis. As modern PDRs use ever narrower pulses, the idea that there is a response region what is nearly a steady-state oscillations as a function of changes in flight path is not quite compelling, but this notation is widely used and has been used for a long time. We can make this notation more precise by defining I to be the in-phase response of the PDR and Q to be

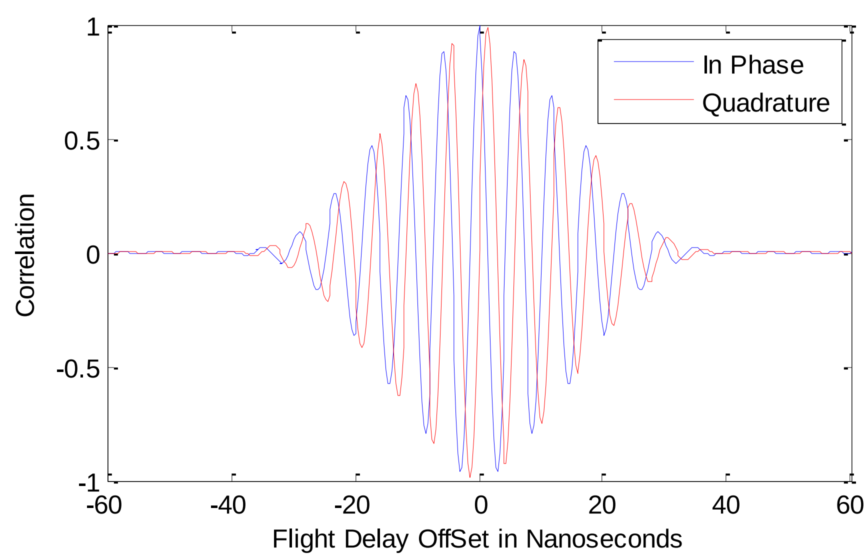

Figure 5. The correlation functions resulting from the waveforms shown in Figure 4.

the quadrature response. We then arbitrarily define $C := I + i \cdot Q$ to be the total response of the PDR. Then the value of C corresponding to the PDR responses as the range is varied through the window of detectability is shown in Figure 6.

When the target is very far away the response is zero. When the target is close enough to induce a response the response as a function of range is a *phase spiral* of gradually increasing amplitude. When the target range exactly matches the time delay between the transmitted signal and the reference signals, the amplitude is maximized and C is real-valued. If the target moves still closer, than the response is a phase spiral of gradually decreasing amplitude. Finally when the target is too close to the radar to be detected the response is negligible or even zero.

The Use of Phase Information

PDRs are not designed to measure range. The simplest, and perhaps historically first, use of PDRs is/was as simple motion detectors; they produce a significantly non-zero response whenever a target is moving somewhere in the field of view. A more sophisticated use of PDRs is to use the Doppler frequency to provide information about the speed of the target's motion. An even more sophisticated use of the PDR is to analyze not only the dominate Doppler Frequency, but also the side lobe structure or the harmonic structure of the response in order to determine more about the nature of the target motion.

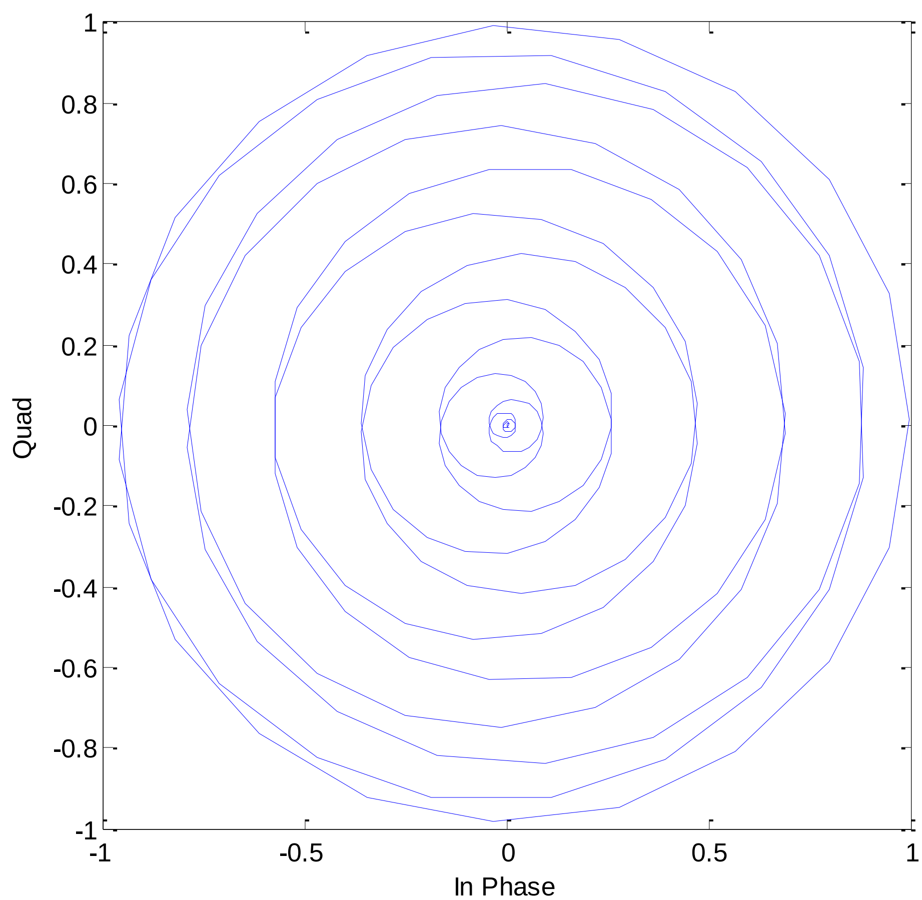

Figure 6. The complex representation of the correlation responses as a function of target range offset.

Directly using phase information is an extreme form of this last strategy. That is changes in the phase information provide the sample-by-sample form of information about the targets motion. Since changes in the phase are used to measure changes in range it is useful to start by understanding the relationship between the phase of instantaneous measurements and the range to the targets, but it is important to understand that the phase information doesn't allow a PDR to directly measure range.

This is because the phase information yields ambiguous range information. To see this consider that at any instance in time the output of the PDR yields a point on the complex plain. If the Radar Cross Section (RCS) of the target is precisely calibrated we could project the measurement onto the appropriate version of the curve shown in Figure 6, and thus estimate the range to the target. However, this is not practical because:

- In most scenarios the uncertainty in the target's Radar Cross Section (RCS) is large.
- Higher quality PDRs strive to minimize the variability of the amplitude response as a function of range over some defined field of view.

As a result the operationally practical model for most PDRs is that all of the instantaneous range information is captured in the phase of the output.

However, if we consider the mapping from the complex plain to the phase it is not a smooth function; it must have what is called a *cut*. In the conventional definitions of phase this cut occurs along the negative portion of the real line. This is illustrated in Figure 7.

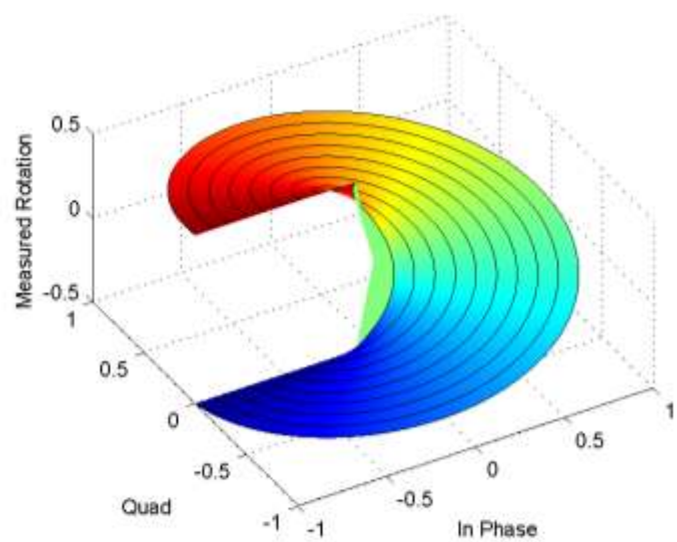

Figure 7. The typical cut in the complex plain used when measuring phase.

To choose to place the cut along the negative real line is arbitrary, but we must place a cut along some ray emanating from the origin.

The implication of this cut is that as a target moves steadily towards the radar the measured phase steadily increases until it reaches π radians, at which point it abruptly jumps to $-\pi$ radians and repeats the process. This can be visualized as a point traversing the corkscrew spiral in 7 at a constant radius and dropping off the upper edge of the gap onto the lower edge of the gap.

This cut produces ambiguity in the range corresponding to a given phase measurement. Specifically, it is not known how many times target has circled the origin. This ambiguity can be more directly visualized by plotting the measured phase (i.e., assuming the standard cut) as a function of the range to the target, as is done in Figure 8. This figure illustrates that a given phase measurement (e.g., 0.2 rotations in the figure) precisely constrains the path length to a set of values that are periodic, with a period equal to the wavelength of the radar. Conceptually, a phase

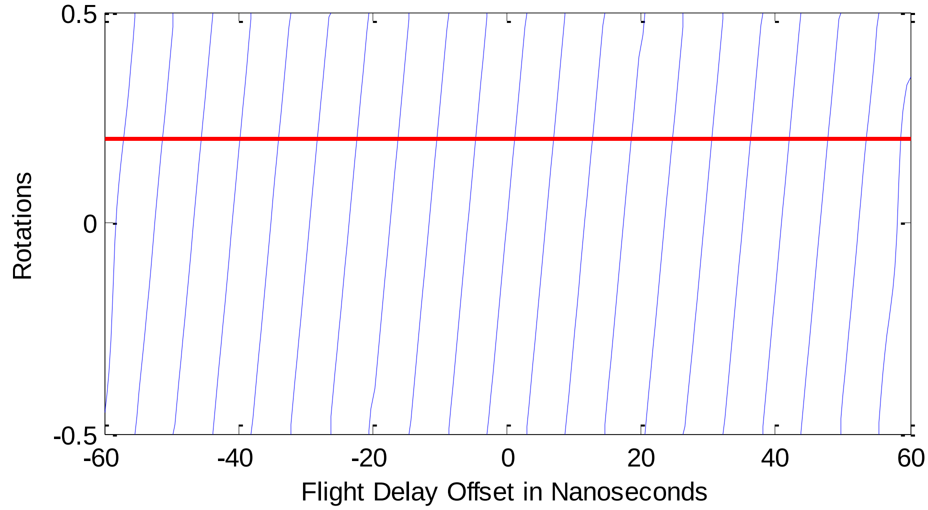

Figure 8. Range ambiguity associated with phase measurements.

measurements doesn't determine the range, but rather determines that the range is one of the values in a set of ranges.

Another way of thinking about this is that the phase information provides local range information, on a scale of less than a wavelength, but provides no global range information. So if the range is approximately known, i.e., to within a fraction of a wavelength, then the phase information provides very precise range information. However, if the range to the target is only weakly known then the phase information in not sufficient to determine the range (essentially at all).

More typically the phase information is not used to determine the range, but rather changes in the phase are used to determine changes in range. If the phase is measured often enough so that the change in range between consecutive measurements is known to be less than half a wavelength then the pair of phase measurements uniquely determines the change in range. A long sequence of phase measurements can be analyzed as a sequence of measurement pairs in order to determine the changes in range between each pair of measurements. That is, small changes in range over short durations can then be accumulated to measure larger changes in range (i.e., multiple wavelength changes) that occur over longer time intervals. This process is called *phase unwrapping* and will be discussed in the next section.

Phase Unwrapping Methods

In mathematical terms: let ϕ be defined as the measured phase, also known as the *wrapped phase*, let r be defined as the flight path length, and let λ be defined as the wavelength. Notice that for the slightly simplified model of the PDR presented above the wrapped phase is,

$$\phi(t) = \mathrm{mod}\left(\frac{r(t) - w/2}{\lambda} + \pi, 2\pi\right) - \pi \,.$$

We define the unwrapped phase ϕ_u such that,

$$\phi_u(t) = \phi(t) + k(t) \cdot 2\pi \quad k(t) \in Z, \forall t \,,$$

and

$$\lambda \cdot \left(\phi_u(t_1) - \phi_u(t_0)\right) = r(t_1) - r(t_0) .$$

That is, the unwrapped phase should equal the wrapped phase plus or minus some integer multiple of 2π, where this multiple of 2π may vary with time. In addition the unwrapped phase should be such that and change in unwrapped phase corresponds to actual change in target range.

Simple Algorithm

If the phase is measured at times $t_i = j \cdot h$ (for $h > 0$) then we may denote the measured phase at time t_j by $\phi_{m,j}$ and the unwrapped phase as $\phi_{u,i}$. Further define the ideal wrapped phase at time t_j as $\phi_i := 2\pi \cdot r/\lambda$. Note that

$$\phi_{m,i} = \phi_i + \varepsilon_i ,$$

where ε_i denotes measurement error and noise.

Definition 1 (*Nyquist Sampling Rate):* If the output samples are taken often enough so that

$$\left|\frac{r_i - r_{i-1}}{\lambda}\right| < 1/2 \quad \forall i ,$$

this can be shown to be equivalent to sampling above the Nyquist rate.

Theorem 1: If the sampling rate is at least the Nyquist rate and if $\varepsilon_i = 0, \forall i$, it is possible to exactly compute $\phi_{u,i}$ .from the $\phi_{m,i}$. Furthermore the construction is

$$\phi_{u,i} = \phi_{u,i-1} + \mathrm{mod}\left(\phi_{m,i} - \phi_{m,i-1} + \pi, 2\pi\right) + \pi .$$

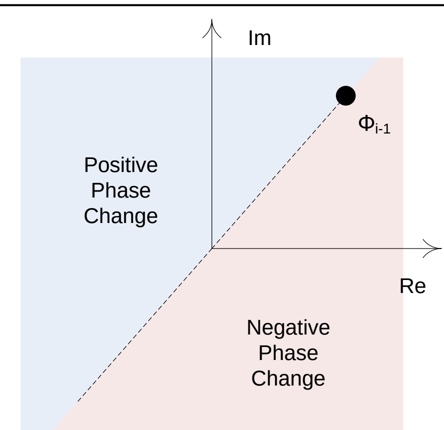

Figure 9. A pictorial representation of the unwrapping formula.

This formula is equivalent to changing the cut for every sample to occur along the ray from the origin in the direction exactly away from the previous sample. This is illustrated in Figure 9. For each sample we construct a line from the sample through the origin and if the next sample is to the right of this line (i.e., when looking at the origin from the current sample) then the change in phase should be interpreted as positive and if it is to the right of the line then the change is negative.

Notice that this algorithm can be implemented in 3 or 4 lines of code in a typical high level language. The complexity is increased somewhat if the implementation must be done in a fixed point, as is the case on a mote, but the extra complexity is not dramatic.

Unwrapping Errors

The phase unwrapping problem becomes interesting in the presence of noise. In this case a sequence of noise events may cause the trajectory in the complex plane, i.e., nominally the phase spirals shown in Figure 6, to appear to wrap around the origin one extra or one fewer times than it really does, i.e., than the noise free measurements would.

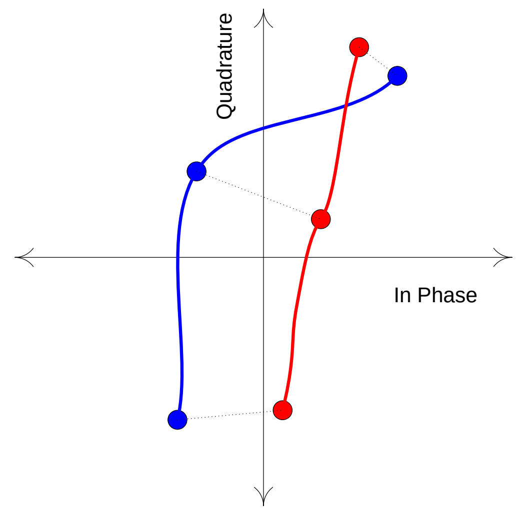

Figure 10. A phase unwrapping error.

This is illustrated in Figure 10. Consider that the blue dots are the ideal radar output and the red dots are the actual outputs, perturbed by noise. Then the blue line is a stylized representation of the ideal complex response over time and the red line is a stylized representation of the estimated complex response as a function of time. Notice that the idealized response crosses the cut ones while the estimated response doesn't. So the simplified algorithm underestimates the change in range by one wavelength.

Notice that phase unwrapping errors can have either sign. Consider the case where the red dots and the blue dots reverse roles, then the simple algorithm would overestimate the change in range by one wavelength. However, phase unwrapping errors are always exactly $\pm 2\pi$.

If the SNR is even modest, e.g., 4 to 5 dB, and the ideal rate of phase change is a small fraction of a rotation, e.g., less than $\pi/6$ radians per sample, then the odds of a phase unwrapping error are very small. However, as the rate of change of the ideal phase approaches $\pm\pi$ radians per sample or as the SNR approaches about -1 dB the error rate of the simple algorithm will approach 50%.

Tracking Algorithm

When considering unwrapping errors between adjacent samples the errors tend to be rare and isolated. However, when considering the total phase change across many samples the result is potentially wrong if any of the pairs of adjacent samples resulted in a phase unwrapping error. It is possible for multiple errors to cancel each other out, but this case is not of conceptual importance. The point is if we start from a given sample and unwrap sequentially over many samples, then each isolated phase unwrapping error has a persistent affect. When unwrapping long sequences of phase measurements even a modest error rate will yield a high probability of some errors and a high probability that the total estimated phase change is distorted by a phase unwrapping error.

This is illustrated in Figure 11. Here we see that the noise produces some random variation about the true phase (in this case zero), but remains "locked" around the true phase until the transition between the 13th and 14th sample, at which point a -2π error is introduced. The affect of this isolated error persists for the rest of the sequence. Because this was an especially noisy example there are several additional phase unwrapping errors in this sequence.

In general, well designed applications should exploit the unwrapped phase information knowing that phase unwrapping errors occur randomly, even if only occasionally.

However, there are some methods that reduce the rate of unwrapping errors. As stated earlier if the rate of change of the ideal phase is low then low pass filtering has the affect of smoothing the noise over many samples, reduces the impact of noise. This doesn't work if the ideal phase is changing faster than limits of the low pass filter. Of course if the rate of change of the ideal phase is steadily changing then it is possible to reduce the noise by fitting a smooth function to the unwrapped phase. This has the affect of averaging noise over many samples, without requiring the phase to be constant; it must only fit well to the selected functional form.

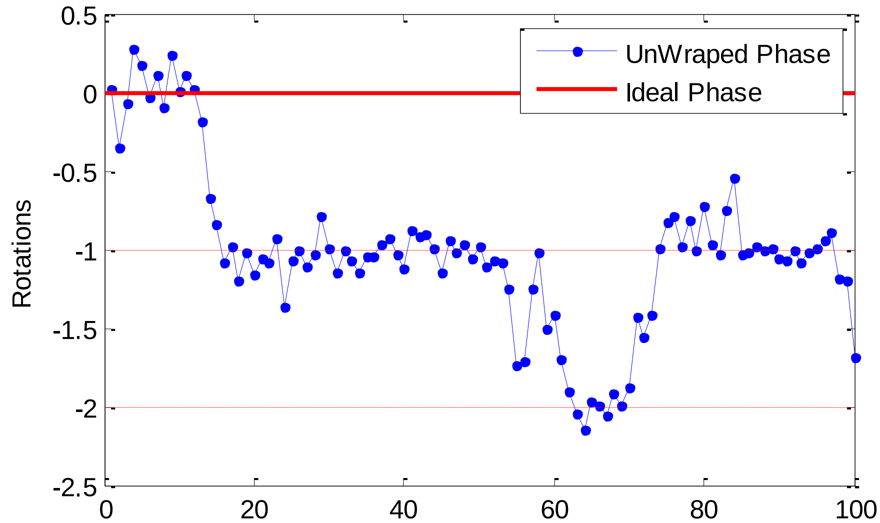

Figure 11. The result of unwrapping a high-noise sequence of phase measurements.

For many applications we have found that fitting the unwrapped phase to short quadratic in order to estimate the expected location of the next sample and then using this estimate to decide the number of rotations around the origin that should be used when unwrapping the next sample, reduces the error rate from about 1 in 100 samples to about 1 in 1000 samples for data from the BumbleBee radar. This procedure is stated more precisely in Table 1**.**

Multiple Targets

Up to this point we have assumed one dominate target within the field of view. In practice there are times when more than one operational target are within view of the same radar and many times when the view of an operational target is corrupted by the presence of other moving objects. In these cases the output of the PDR is the sum of the two returns.

Specifically, if $c_i(t)$ for $i \in \{1 \ldots n\}$ represents the complex returns from each of the n targets, then the output of the radar will be

$$C(t) = \sum_{i=0}^{n-1} c_i(t) + \varepsilon,$$

where ε is some noise function. At first glance disentangling the phase information from multiple targets may seem hopeless, but a more carful look reveals that the in most cases one of the targets will dominate the phase information. In order to better understand this, consider the somewhat stylized case where a collection of targets present a constant RCS over a fairly long

Table 1. Pseudo-code for fitting a quadratic to the phase data in order to unwrap the phase as part of an unwrapped phase tracking algorithm

```
UnPhse(0) = Phase(0);
for (i = 1; i++; i < N) {
  Min = min(0, i-30);
  Quad = Fit([Low, High-1], UnPhase[Min : i-1]);
  Est = Extrapolate(Quad, High);
  UnPhase(i) = Est + mod(Phase(i) - Est + Pi, 2*Pi) - Pi
}
```

time interval. In this case each complex return can be modeled by

$$c_i(t) = \frac{k_i}{r_i(t)^2} \cdot \exp\left(2\pi \cdot r_i(t)\right).$$

Here k encapsulates the affects of the RCS and the gain of the various radar elements and r represents the flight path length. The large natural spread of RCSs combined with the $1/r^2$ dependence flight path length tends to make coefficient of the exponent, i.e., k/r^2, spread over a couple of orders of magnitude. That is the largest return will typically be two or three or more times as large as the second largest return. In this case the phase of the total return is a slight perturbation of the phase of the largest return.

This is illustrated in Figure 12. In this example the strong signal is 4 times as large as the week signal. The phase information of the combined signal is equivalent to a slight periodic stretching of the phase information that would be presented by the dominant target alone.

This insight leads to the following useful model. The phase of the total return will almost always be the phase resulting from strongest/nearest target with a small perturbation caused by other further/weaker targets. The exception occurs when two targets are very nearly the same strength and same distance from the radar.

Target Coherence

Finally, some targets, like the human body, do not follow the model presented in the previous section. They represent a collection of many small returns. The well worn joke that a particular plane is a collection of parts flying in close formation is especially application to radar returns. The actual returns are typically generated off of individual edges, dimples, protrusions, and

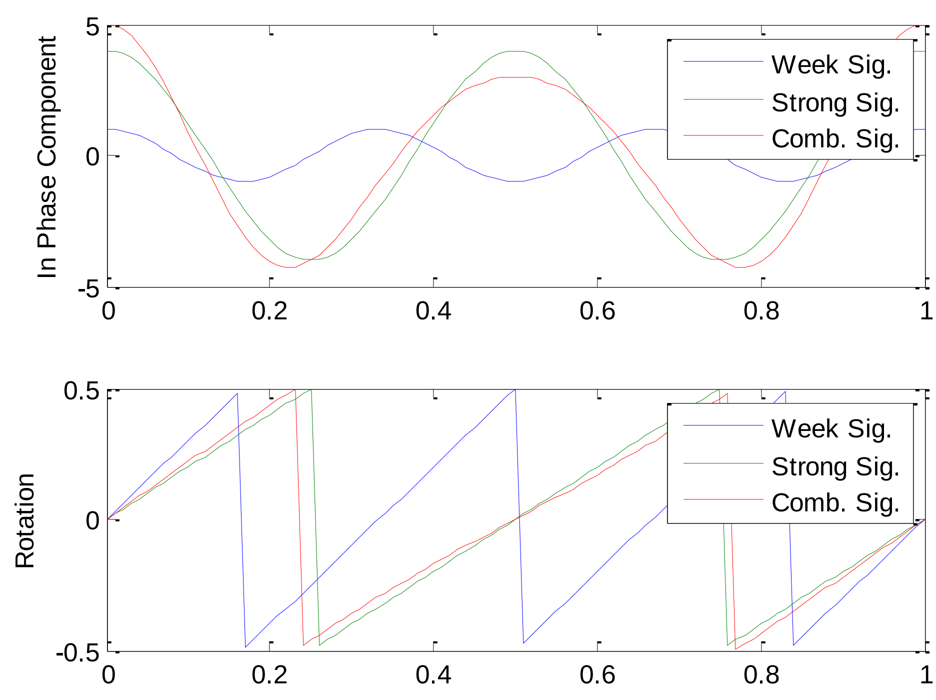

Figure 12. The affect of mixing signals of unequal strength on phase measurements.

assorted other features. On so called "soft targets" these individual return generating features tend to move with respect to each other as the whole body moves through the field of view.

For example a human walking might generate significant returns from the teeth, torso, arms, belt buckle, legs, and shoes. All of these objects move with respect to each other in a rather complicated pattern as the person walks through the field of view.

Figure 13. The concept of effective center of return for a cluster of closely connected returns.

This is illustrated in Figure 13. Here we imagine the trajectory of the top three sources of return from a soft target (i.e., shown in three colors). The result is a complicate pattern, however, at any given time the combination of these three returns may be equivalent to a return from a virtual target following the trajectory shown by the black line.